Formal Proof Theory & Connections to Model Theory

Goals

1. Introduce and build up Formal Proof Theory
2. Connect Formal Proof Theory to Model Theory (the Completeness Theorem)
3. Discuss more foundations of Model Theory.

1 & 2 are very related, and 3 is important in overall continuing our discussion of Model Theory and setting up material for future talks.

Model Theory vs. Formal Proof Theory

(Definitions from Wikipedia)

- **Model Theory**: The study of the relationship between formal theories (collections of sentences in a formal language expressing statements about a mathematical structure), and their models, taken as interpretations that satisfy the sentences of that theory.

- **Formal Proof Theory**: A branch of mathematical logic that represents proofs as formal mathematical objects.

As we will see, the two are very interconnected.

Setting up Formal Proof Theory : Step-by-Step

Step 0) • Goals: Develop a system of representing proofs as formal mathematical objects that is easy to define and analyze mathematically.

• Sacrificing: Ease of use in practical settings.

Step 1) • We build our proof theory on one rule of inference : Modus Ponens

↳ (takes in premises & returns conclusions)

$$\frac{\varphi \qquad \varphi \to \psi}{\psi}$$

Informally, Modus Ponens means:
"If we have proved both φ and $\varphi \to \psi$, then we can conclude ψ."

Step 1 (continued) Modus Ponens is so fundamental to our definition of formal proofs that we formally embed it in our definition of formal proofs.

Step 2) We now single out some "obviously valid" statements and call them "logical axioms."

Definition: A logical axiom of $\mathcal{L}$ is any sentence of $\mathcal{L}$ that is a universal closure of a formula of one of the types we will list below.

First, some Review and New Terminology ...

Model Theory Mini-Review

Definition: A lexicon $\mathcal{L}$ for predicate logic consists of a set $\mathcal{L}$ (of non-logical symbols, i.e. not $\wedge, \vee, \neg, \rightarrow, \leftrightarrow, \forall, \exists$, or $=$), partitioned into disjoint sets: $\mathcal{L} = \mathcal{F} \cup \mathcal{P}$.

- $\mathcal{F}$ = set of function symbols.

  (function: given objects, produces an object, e.g. $1+2 \rightarrow 3$)
- $\mathcal{P}$ = set of predicate symbols.

  (predicate: given objects, produces a claim, e.g. $1<2$ → gives you the claim that 1 is smaller than 2; and, the claim $1<2$ is not an object from the domain.)

Model Theory Mini-Review

Definition: A sentence of $\mathcal{L}$ is a Boolean-valued well-formed formula (i.e. a finite sequence of symbols from a given alphabet that is part of formal language $\mathcal{L}$) with no free variables.

Definition: If φ is a formula, a universal closure of

Φ is any sentence of the form

$\forall x_1 \forall x_2 \cdots \forall x_n \, \varphi$ where $n \geq 0$.

$\forall$ for all free variables of φ (non-$\forall$, non-$\exists$ in φ already).

New Terminology

Definition: A truth assignment is a set of basic formulas (formulas that do not begin with a logical connective in Polish notation) of $\mathcal{L}$ into $\{0,1\} = \{F,T\}$.

Given such a v, we define (recursively) $\bar{v}(\varphi) \in \{F,T\}$ as follows:

1. $\bar{v}(\neg\varphi) = 1 - \bar{v}(\varphi)$
2. $\bar{v}(\wedge\varphi\psi)$, $\bar{v}(\vee\varphi\psi)$, $\bar{v}(\rightarrow\varphi\psi)$, and $\bar{v}(\leftrightarrow\varphi\psi)$

are obtained from $\bar{v}(\varphi)$ and $\bar{v}(\psi)$ using the truth tables for $\vee, \wedge, \rightarrow, \leftrightarrow$.

Definition: φ is a propositional tautology iff $\bar{v}(\varphi) = T$ for all truth assignments v.

Back to defining Logical Axioms

Definition: A logical axiom of $\mathcal{L}$ is any sentence of $\mathcal{L}$ that is a universal closure of a formula of one of the types we will list below:

1. Propositional Tautologies
2. $\varphi \rightarrow \forall x \varphi$ where x is not free in φ.

(3) $\forall x(\varphi \to \psi) \to (\forall x\, \varphi \to \forall x\, \psi)$

(4) $\forall x\, \varphi \to \varphi(x \rightsquigarrow \tau)$

(5) $\varphi(x \rightsquigarrow \tau) \to \exists x\, \varphi$

where τ is any term that is free for x in τ, and $x \rightsquigarrow \tau$ means "assign x to be τ"

(6) $\forall x \neg\varphi \leftrightarrow \neg\exists x\, \varphi$

(7) $x = x$

(8) $x = y \leftrightarrow y = x$

(9) $x = y \wedge y = z \to x = z$

(10) $x_1 = y_1 \wedge \cdots \wedge x_n = y_n \to (f(x_1 \ldots x_n) = f(y_1 \ldots y_n))$,

whenever $n > 0$ and f is an n-place function symbol of $\mathcal{L}$.

(11) $x_1 = y_1 \wedge \cdots \wedge x_n = y_n \to (p(x_1 \ldots x_n) \leftrightarrow p(y_1 \ldots y_n))$,

whenever $n > 0$ and p is an n-place predicate symbol of $\mathcal{L}$.

Logical Axioms are Logically Valid

Remark: All the logical axioms are logically valid.

Definition: If ψ is a formula of $\mathcal{L}$, then ψ is **logically valid** iff $\mathcal{U} \models \psi[\sigma]$ for all $\mathcal{L}$-structures $\mathcal{U}$ and all assignments σ for ψ in $\mathcal{U}$.

↳ Informally: "It's impossible for our logical axioms to take a form where the premises are true and the condition is false."

Next Step in Defining our Formal Proof Theory

Step 3) Defining a formal proof.

Definition: If Σ is a set of sentences of $\mathcal{L}$ (where a sentence is a Boolean-valued well-formed formula, i.e. a finite sequence of symbols, with no free variables), then a formal proof from Σ is a finite non-empty sequence of sentences of $\mathcal{L}$, $\varphi_0, \dots, \varphi_n$, such that for each i, either:

① $\varphi_i \in \Sigma$, ② φ_i is a logical axiom, or

③ For some $j, k < i$, φ_i follows from φ_j, φ_k by Modus Ponens (that is, φ_k is $(\varphi_j \to \varphi_i)$).

This sequence is a formal proof of its last sentence, φ_n.

Notice: Our definition of a formal proof is entirely based on the sentences of $\mathcal{L}$, logical axioms, and our one rule of inference (Modus Ponens).

Notation

- If Σ is a set of sentences of $\mathcal{L}$, and φ is a sentence of $\mathcal{L}$, then $\Sigma \vdash_{\mathcal{L}} \varphi \iff$ there exists a formal proof of φ from Σ

Examples of Formal Proofs

Example #1: (a) Show: $p \wedge q \vdash p$.

(b) Our sentences: $p \wedge q$, i.e. $\Sigma = \{p \wedge q\}$ and $\mathcal{L}$ contains at least the proposition letters p and q.

(c) What we want to prove: p.

(Example, continued).

Our formal proof:

(0)	$p \wedge q \to p$	tautology
(1)	$p \wedge q$	given
(2)	p	0, 1, modus ponens.

The formal proof itself consists of the sequence of 3 sentences, $(p \wedge q \to p,\ p \wedge q,\ p)$, not the commentary.

Remark: Given any sequence of sentences $(\varphi_0, \dots, \varphi_n)$, without any commentary, one can determine whether it forms a formal proof, since we may, for each φ_i, check all 13 possible justifications for φ_i being correct (the 11 types of logical axioms plus Modus Ponens plus $\varphi_i \in \Sigma$).

Aligning Formal Proof Theory and Model Theory

Let's ensure that our formal proof theory aligns with our model theory.

Lemma: (Soundness) If $\Sigma \vdash_{\mathcal{L}} \varphi$, then $\Sigma \models \varphi$.

- Recall that: (a) $\Sigma \vdash_{\mathcal{L}} \varphi$ means that for sentences Σ of $\mathcal{L}$, there exists a formal proof of φ from Σ.

(b) $\Sigma \models \varphi$ means for structure $\mathfrak{A}$ such that $\mathfrak{A} \models \Sigma$, $\mathfrak{A} \models \varphi$, where $\mathfrak{A} \models \varphi$ means φ is true in $\mathfrak{A}$, where $\mathfrak{A}$ is a structure for $\mathcal{L}$ and φ is a sentence of $\mathcal{L}$.

(Soundness, continued).

Recall the **Definition:** In mathematical logic, a structure consists of a set along with a collection of finite-arity operations and finite relations that are defined on it.

So, Soundness (If $\Sigma \vdash_{\mathcal{L}} \varphi$, then $\Sigma \models \varphi$) means:

If we can prove φ from sentences Σ of $\mathcal{L}$, then in (any) $\mathcal{L}$-structure $\mathfrak{A}$ in which Σ is true, φ is also true.

Aligning Formal Proof Theory and Model Theory: Completeness

Lemma: (Completeness) If $\Sigma \models \varphi$, then $\Sigma \vdash_{\mathcal{L}} \varphi$.

Important Note: Our definition of formal proof is very simple, except for the list of logical axioms. The choice of which statements to include on this list is a bit arbitrary, and differs in different texts.

There are only three important things about this list....

The Three Important Things about our Logical Axioms List

1. Every logical axiom is logically valid, so that Soundness is true.
2. We have listed enough logical axioms to verify Completeness.
3. When $\mathcal{L}$ is finite, the set of logical axioms is decidable.

Decidability

- When $\mathcal{L}$ is finite, we may view syntactic objects as possible inputs into a computer.
- Decidable means that a computer can check whether or not a sequence of formulas is a formal proof, and the computer can in principle generate its own formal proofs.

Some Strategies for Constructing Proofs

We will now establish a few general principles that show how informal mathematical arguments can be replicated in formal proof theory.

Lemma (The Deduction Theorem)

$$\Sigma \vdash_{\mathcal{L}} (\varphi \to \psi) \quad \text{iff} \quad \Sigma \cup \{\varphi\} \vdash_{\mathcal{L}} \psi.$$

↳ Tells us: To prove $\varphi \to \psi$, we may assume that φ

is true and derive ψ.

Consistency

<u>Definition:</u> If Σ is a set of sentences of $\mathcal{L}$, then Σ is syntactically inconsistent ($\neg Con_{\vdash,\mathcal{L}}(\Sigma)$) iff there is some sentence φ of $\mathcal{L}$ such that $\Sigma \vdash_{\mathcal{L}} \varphi$ and $\Sigma \vdash_{\mathcal{L}} \neg\varphi$.

"consistent" ($Con_{\vdash,\mathcal{L}}(\Sigma)$) means "not inconsistent".

Compare this to our semantic (model theory) notion of consistency:

<u>Definition:</u> If Σ is a set of sentences in $\mathcal{L}$, then Σ is semantically consistent (or satisfiable) (i.e. $Con_{\models}(\Sigma)$) iff there is some $\mathcal{L}$-structure $\mathcal{U}$ such that $\mathcal{U} \models \Sigma$.

"inconsistent" ($\neg Con_{\models}(\Sigma)$) means "not consistent".

Some Strategies for Constructing Proofs (Continued)

<u>Lemma:</u> If Σ is a set of sentences of $\mathcal{L}$, then the following are equivalent:

(a) $\neg Con_{\vdash,\mathcal{L}}(\Sigma)$

(b) $\Sigma \vdash_{\mathcal{L}} \psi$ for <u>all</u> sentences ψ of $\mathcal{L}$.

Lemma: (Proof by Contradiction) If Σ is a set of (consistent) sentences of $\mathcal{L}$ and φ is a set of sentences of $\mathcal{L}$, then:

① $\Sigma \vdash_{\mathcal{L}} \varphi$ iff $\neg \mathrm{Con}_{\vdash, \mathcal{L}} (\Sigma \cup \{\neg\varphi\})$.

② $\Sigma \vdash_{\mathcal{L}} \neg\varphi$ iff $\neg \mathrm{Con}_{\vdash, \mathcal{L}} (\Sigma \cup \{\varphi\})$.

Proof idea: Follows from definition of $\neg\mathrm{Con}$ and by the Deduction Theorem.

(More Strategies)

Lemma: (Tautological Reasoning) If $\psi, \varphi_1, \dots, \varphi_n$ are sentences of $\mathcal{L}$ and ψ follows tautologically from $\varphi_1, \dots, \varphi_n$ (i.e. $(\varphi_1 \wedge \dots \wedge \varphi_n) \to \psi$ is a propositional tautology), then $\{\varphi_1, \dots, \varphi_n\} \vdash_{\mathcal{L}} \psi$.

↳ Proof: Note that $\varphi_1 \to (\varphi_2 \to (\cdots \to (\varphi_n \to \psi) \cdots))$ is a tautology, and use Modus Ponens n times.

Lemma: (Transitivity of $\vdash$) If $\{\varphi_1, \dots, \varphi_n\} \vdash_{\mathcal{L}} \psi$, and $\Sigma \vdash_{\mathcal{L}} \varphi_i$ for $i = 1, \dots, n$, then $\Sigma \vdash_{\mathcal{L}} \psi$.

Quantifier Rules

Now, we come to some rules for handling quantifiers ($\forall, \exists$, etc.)
In informal mathematical reasoning, quantifiers are usually not written explicitly, but handled implicitly by the informal analog of the following.

In the following rules:

- Let τ be a term ("arbitrary term") with no variables, so $\varphi(\tau)$ is a sentence.
- Let c be a constant not in $\mathcal{L}$, and $\mathcal{L}' = \mathcal{L} \cup \{c\}$
- Let ψ be a sentence of $\mathcal{L}$.

Quantifier Rule #1: Universal Instantiation

- Informal: If a universal statement $\forall x\, \varphi(x)$ is true, then we can conclude a specific instance $\varphi(\tau)$.
- Formal: $\forall x\, \varphi(x) \vdash_{\mathcal{L}} \varphi(\tau)$.

Quantifier Rule #2: Universal Generalization

- Informal: If $\varphi(c)$ is true for all arbitrary c, then $\forall x\, \varphi(x)$ is true.
- Formal:

$$\frac{\Sigma \vdash_{\mathcal{L}'} \varphi(c)}{\Sigma \vdash_{\mathcal{L}} \forall x\, \varphi(x)}.$$

Quantifier Rule # 3: Existential Instantiation

Informal: Assume for a (fixed) c, $\varphi(c)$ is true, and suppose sentences Σ of $\mathcal{L}$ are true. If this implies ψ is true, then we know that if Σ are true sentences and if $\exists x$ such that $\varphi(x)$ is true, then this implies ψ is true.

Formal:
$$\frac{\Sigma \cup \{\varphi(c)\} \vdash_{\mathcal{L}'} \psi}{\Sigma \cup \{\exists x \; \varphi(x)\} \vdash_{\mathcal{L}} \psi}$$

Quantifier Rule # 4: Existential Generalization

Informal: If we can prove that φ holds of a specific τ, then we know $\exists x \, \varphi(x)$.

Formal: $\varphi(\tau) \vdash_{\mathcal{L}} \exists x \, \varphi(x)$.

Example of using our Quantifier Rules and other New Tools

Prove: $\exists x \, \forall y \; p(x,y) \rightarrow \forall y \; \exists x \; p(x,y)$.

Informal: Assume $\exists x \, \forall y \, p(x,y)$ and fix (Existential Instantiation) c such that $\forall y \; p(c,y)$. Consider any object d. Then $p(c,d)$ by Universal Instantiation. So, $\exists x \; p(x,d)$ by Existential Generalization. But d was arbitrary (Universal Generalization), so

$\forall y \, \exists x \; p(x,y).$

Example (continued)

Formal Proof: [The order of application of the rules gets permuted, but the idea stays the same.]

(0)	$p(c,d) \vdash_{\mathcal{L}''} p(c,d)$		Tautology
(1)	$p(c,d) \vdash_{\mathcal{L}''} \exists x \, p(x,d)$		0, EG
(2)	$\forall y \, p(c,y) \vdash_{\mathcal{L}''} \exists x \, p(x,d)$		1, UI
(3)	$\forall y \, p(c,y) \vdash_{\mathcal{L}'} \forall y \, \exists x \, p(x,y)$		2, UG
(4)	$\exists x \, \forall y \, p(x,y) \vdash_{\mathcal{L}} \forall y \, \exists x \, p(x,y)$		3, EI
(5)	$\emptyset \vdash_{\mathcal{L}} \exists x \, \forall y \, p(x,y) \to \forall y \, \exists x \, p(x,y)$		Deduction Theorem

The Completeness Theorem: Motivation

Motivation: (1) This result relates the semantic ($\models$) notions to the syntactic ($\vdash$) notions. [It may be viewed either as a result about consistency or as a result about provability.]

(2) The Completeness Theorem is a main result in the direction of demonstrating that our $\vdash$ has the properties of an expected notion of provability

The Completeness Theorem : Statement

Theorem: (Completeness Theorem) Let Σ be a set of sentences of $\mathcal{L}$. Then:

1. $Con_{\models}(\Sigma)$ iff $Con_{\vdash,\mathcal{L}}(\Sigma)$.
2. For every sentence φ of $\mathcal{L}$, $\Sigma \models \varphi$ iff $\Sigma \vdash_{\mathcal{L}} \varphi$.

Once this is proved, we can drop the subscripts on the "$\vdash$" and on the "Con."

Main Lemma used in the proof: Let Σ be a set of sentences of $\mathcal{L}$, and assume that $Con_{\vdash,\mathcal{L}}(\Sigma)$. Then, $Con_{\models}(\Sigma)$.

Note: We already have $\Sigma \vdash_{\mathcal{L}} \varphi \Rightarrow \Sigma \models \varphi$, the Soundness Lemma we mentioned from before.

The Completeness Theorem : Proof

- We won't go over the whole proof (it's very involved). But along the way in proving The Completeness Theorem, we develop useful concepts and techniques, which we now will highlight.

Definition: A set of sentences Σ in $\mathcal{L}$ is maximally $(\vdash, \mathcal{L})$ consistent iff

1. $Con_{\vdash, \mathcal{L}}(\Sigma)$ and
2. There is no set of sentences Π in $\mathcal{L}$ such that $Con_{\vdash, \mathcal{L}}(\Pi)$ and $\Sigma \subsetneq \Pi$.

Lemmas about Maximally $(\vdash, \mathcal{L})$-consistent Σ

Lemma: If Δ is a set of sentences in $\mathcal{L}$ and $Con_{\vdash, \mathcal{L}}(\Delta)$, then there is a Σ in $\mathcal{L}$ such that $\Sigma \supseteq \Delta$ and Σ is maximally $(\vdash, \mathcal{L})$ consistent.

Lemma: Assume that Σ in $\mathcal{L}$ is maximally $(\vdash, \mathcal{L})$ consistent. Then for any sentence φ, ψ of $\mathcal{L}$:

1. $\Sigma \vdash_{\mathcal{L}} \varphi$ iff $\varphi \in \Sigma$
2. $(\neg\varphi) \in \Sigma$ iff $\varphi \notin \Sigma$
3. $(\varphi \vee \psi) \in \Sigma$ iff $\varphi \in \Sigma$ or $\psi \in \Sigma$.

↳ **Idea:** A maximally $(\vdash, \mathcal{L})$ consistent set of sentences Σ in $\mathcal{L}$ "contains all that sentences it possibly can contain."

A Note about the Completeness Theorem's Proof

In the proof of the Completeness Theorem, we build a model out of the terms of the language. The use of such symbolic expressions as mathematical objects has its roots in the algebra of the 1800s.

As an example: the ring $F[x]$ of polynomials over a field F, and the use of this ring for obtaining algebraic extensions of F.

More Foundations of Model Theory

We now transition to discussing more foundations of Model Theory. This part of the talk will not relate as closely to Formal Proof Theory, but is important in furthering our conversation on Model Theory and setting up material that is useful for future talks.

We begin by discussing Complete Theories.

Complete Theories

The notion of a complete set of axioms is completely unrelated to the "complete" in the "Completeness Theorem." It is closely related to the notion of elementary equivalence.

Definition: If Σ is a set of sentences of $\mathcal{L}$, then Σ is complete (with respect to $\mathcal{L}$) iff Σ is semantically consistent and for all sentences φ of $\mathcal{L}$, either $\Sigma \models \varphi$ or $\Sigma \models \neg\varphi$

Elementary Equivalence

Definition: If $\mathfrak{A}$, $\mathfrak{B}$ are structures for $\mathcal{L}$, then $\mathfrak{A} \equiv \mathfrak{B}$ ($\mathfrak{A}$, $\mathfrak{B}$ are elementarily equivalent) iff for all $\mathcal{L}$-sentences φ: $\mathfrak{A} \models \varphi$ iff $\mathfrak{B} \models \varphi$.

Lemma: If Σ is a set of sentences of $\mathcal{L}$, then Σ is complete iff Σ is consistent and $\mathfrak{A} \equiv \mathfrak{B}$ whenever $\mathfrak{A}$ and $\mathfrak{B}$ are models of Σ.

Isomorphic models are elementarily equivalent

Definition: $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic ($\mathfrak{A} \cong \mathfrak{B}$) iff there exists an isomorphism from $\mathfrak{A}$ onto $\mathfrak{B}$.

(Not in full detail) ϕ is an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ iff $\phi : A \xrightarrow[\text{onto}]{1-1} B$ (i.e. bijection) and ϕ preserves the structure.

Lemma: If $\mathfrak{A} \cong \mathfrak{B}$, then $\mathfrak{A} \equiv \mathfrak{B}$

(i.e isomorphic models are elementarily equivalent).

The Łoś – Vaught Test: κ-categorical

Now, let's describe a method for proving that a given set of sentences Σ is complete: The Łoś – Vaught Test. It involves the notion of categoricity.

Definition: Suppose that Σ is a set of sentences of $\mathcal{L}$ and κ any cardinal. Then Σ is κ-categorical iff all models of Σ of size κ are isomorphic.

The Łoś – Vaught Test : Statement

Theorem: (The Łoś – Vaught Test) Let Σ be a set of sentences of $\mathcal{L}$, and assume:

1. Σ is consistent.
2. All models of Σ are infinite.
3. $\mathcal{L}$ is countable.
4. Σ is κ-categorical for some infinite κ.

Then, Σ is complete.

end of content

Summary

1. We introduced and built up Formal Proof Theory step-by-step:
   - Goal: striving for a system of representing proofs as formal mathematical objects that is easy to define and analyze mathematically.
   - Want to also establish general principles that show how informal mathematical arguments can be replicated in formal proof theory.

(2) We connected Formal Proof Theory to Model Theory

- We showed $\Sigma \models \varphi$ iff $\Sigma \vdash_{\mathcal{L}} \varphi$ and $Con_{\models}(\Sigma)$ iff $Con_{\vdash,\mathcal{L}}(\Sigma)$, ensuring our formal proof theory aligns with our model theory. This is the Completeness Theorem, which demonstrates that our $\vdash$ has the properties of an expected notion of provability.

(3) We developed more foundations of Model Theory.

- Complete set of sentences Σ
- Elementary equivalent models
- κ-categorical set of sentences Σ
- The Łoś-Vaught Test for proving that a given set of sentences Σ is complete.

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